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# Multi-component bi-super Hamiltonian KdV systems

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**Abstract**
It is shown that a new class of classical multi-component super KdV equations is bi-super Hamiltonian by extending the method for the verification of graded Jacobi identity. The multi-component extension of super mKdV equations is obtained by using the super Miura transformation.

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## 1. Introduction

The theories of infinite-dimensional super integrable systems have drawn a lot of attention in the last two decades, for example, see [1, 2]. Research on the classical multi-component integrable systems has also become quite active more recently [3–9]. In this work we construct an extension of classical multi-component Korteweg de Vries (KdV) system to multi-component super integrable systems by employing a bi-super Hamiltonian formalism. Such systems are called super because they contain both bosonic and fermionic fields. However, there is no supersymmetry transformation between the fields as known from the one-component super KdV equations [10–12]. On the other hand, there also exist supersymmetric extensions of KdV equation, namely, there exist supersymmetry transformations but these bi-super Hamiltonian systems have a non-local nature [13–15].

We first introduce skew-symmetric super Hamiltonian operators. It is shown that they satisfy the graded Jacobi identity by using the method of prolongation [12, 16]. The set of multi-component super integrable partial differential equations is derived by introducing associated super Hamiltonians. Furthermore, introducing super Miura transformation, a multi-component super extension modified Korteweg de Vries (mKdV) system is obtained. The paper is organized as follows. In section 2 we investigate the properties of super Hamiltonian operators. It is shown that the second Hamiltonian operator satisfies the graded Jacobi identity by means of constraint between the constant parameters of the system while the first does trivially. In section 3 the multi-component super evolution equations are derived from super Hamiltonians . In section 4 we obtain the multi-component super mKdV equations by using multi-component super Miura transformation. It is observed that the new systems are reduced to well known systems of one-component super evolution equations and in the vanishing fermionic fields limit we get the multi-component and one component corresponding KdV systems.

## 2. The super Hamiltonian operators and Jacobi identity

This section is devoted to the study of properties of super Hamiltonian operators. We first consider a set of fields $\phi_A$ which contains both commuting and anti-commuting fields, such as

$$\phi_A = \begin{pmatrix} u_\alpha \\ \xi_a \end{pmatrix} \tag{1}$$

where $u_\alpha(x,t)$ is assumed to be a commuting (bosonic) field while $\xi_a(x,t)$ is an anti-commuting (fermionic) field in 1+1 dimensions, $\alpha = 1, 2, \ldots, m$ and $a = 1, 2, \ldots, n$. A $Z_2$ grading is introduced such that $\tilde{p}(\phi)$ is equal to 0 if $\phi_A$ is commuting or 1 if it is anti-commuting.

The evolution equation of a continuous dynamical Hamiltonian system is given by

$$\partial_t \phi_A = \sum_B J_{AB} \frac{\delta \boldsymbol{H}}{\delta \phi_B} = \sum_B J_{AB} E_B(H) \tag{2}$$

where $E_B$ is the Euler operator,

$$E_B = \sum_{k=0}^{\infty} (-\partial_x)^k \frac{\partial}{\partial_x^k \phi_B} \tag{3}$$

and $J$ is a certain differential operator and $\boldsymbol{H}$ is a suitable functional. Functionals are defined as modulo the integral of total derivative terms as

$$\boldsymbol{F} = \int F[\phi_A]\,\mathrm{d}x \tag{4}$$

where $F[\phi_A]$ is the element of the algebra of functions of $x$, the fields $\phi_A(x)$ and their derivatives. The operator $J$ defines a Poisson bracket as

$$\{\boldsymbol{F}, \boldsymbol{G}\} = \sum_{AB} \int [J_{AB} E_B(G)] E_A(F)\,\mathrm{d}x. \tag{5}$$

Here the ordering of the arbitrary functionals $\boldsymbol{F}$ and $\boldsymbol{G}$ becomes important for the graded systems. The fundamental Poisson bracket is

$$\{\phi_A(x), \phi_B(x')\} = J_{AB}\delta(x - x') \tag{6}$$

that leads to the following expression for the evolution equation (2):

$$\partial_t \phi_A = \{\phi_A, \boldsymbol{H}\}. \tag{7}$$

$J$ is called a Hamiltonian operator if the Poisson bracket is skew-symmetric as

$$\{\boldsymbol{F}, \boldsymbol{G}\} = -(-1)^{\tilde{p}(\boldsymbol{F})\cdot\tilde{p}(\boldsymbol{G})}\{\boldsymbol{G}, \boldsymbol{F}\} \tag{8}$$

where the grading $\tilde{p}(\boldsymbol{F})$ is equal to 0 (1) if an arbitrary functional $\boldsymbol{F}$ is bosonic (fermionic) and satisfies the Jacobi identity, which can be given as vanishing the prolongation of an evolutionary vector field $v_{J\Theta}$ associated with every Hamiltonian $\boldsymbol{H}$, as follows [12, 16]:

$$\mathrm{pr}\, v_{J\Theta}(I) = 0 \tag{9}$$

where $I$ is the graded cosymplectic functional two-vector given as

$$I = \frac{1}{2} \sum_{A,B} \int J_{AB} \Theta_B \wedge \Theta_A \,\mathrm{d}x. \tag{10}$$

Here the set $\Theta_A = \{\theta_\alpha, \eta_a\}$ forms a basis of bosonic and fermionic uni-vectors, dual to the one-forms $\{u_\alpha, \xi_a\}$, respectively. Note that

$$\Theta_A \wedge \Theta_B = -(-1)^{\tilde{A}.\tilde{B}} \Theta_B \wedge \Theta_A \tag{11}$$

and $\Theta \wedge \Theta \neq 0$ if $\Theta$ is fermionic.

We now introduce the super Hamiltonian operators

$$J^{(1)}_{AB} = \begin{pmatrix} \delta_{\alpha\beta}\partial_x & 0 \\ 0 & \delta_{ab} \end{pmatrix} \tag{12}$$

and

$$J^{(2)}_{AB} = \begin{pmatrix} j_{\alpha\beta} & j_{\alpha b} \\ j_{a\beta} & j_{ab} \end{pmatrix} \tag{13}$$

where

$$j_{\alpha\beta} = b_{\alpha\beta}\partial_x^3 + 2C_{\alpha\beta\gamma}u_\gamma\partial_x + C_{\alpha\beta\gamma}u_{\gamma,x} \tag{14}$$

$$j_{\alpha b} = K_{\alpha bd}\xi_d\partial_x + L_{\alpha bd}\xi_{d,x} \tag{15}$$

$$j_{a\beta} = M_{a\beta d}\xi_d\partial_x + N_{a\beta d}\xi_{d,x} \tag{16}$$

$$j_{ab} = \Lambda_{ab}\partial_x^2 + \Omega_{ab\gamma}u_\gamma \tag{17}$$

where $u_{\alpha,x} = \partial_x u_\alpha$ and all coefficients apart from $u(x, t)$ and $\xi(x, t)$ are constants. It is easy to see that the operator $J^{(1)}_{AB}$ is a Hamiltonian operator because it is skew-symmetric and Jacobi identity is trivially satisfied, there are no variable coefficients in its expression. On the other hand, the second operator $J^{(2)}_{AB}$, which is skew-symmetric, contains $x$ and $t$ dependent coefficients. In order to show that it is a Hamiltonian operator, the graded Jacobi identity should be satisfied. Equation (9) for the second operator becomes

$$\mathrm{pr}\, v_{J\Theta}(I) = \frac{1}{2}\int \mathrm{pr}\, v_{J\Theta}\left(J^{(2)}_{AB}\right)\Theta_B \wedge \Theta_A\, \mathrm{d}x = 0. \tag{18}$$

Here Einstein sum rule is employed and it will be used from now on. Equation (18) can be written as

$$\begin{aligned} \mathrm{pr}\, v_{J\Theta}(I) = \frac{1}{2}\int &[\mathrm{pr}\, v_{J\Theta}(j_{\alpha\beta})\theta_\beta \wedge \theta_\alpha + \mathrm{pr}\, v_{J\Theta}(j_{\alpha b})\eta_b \wedge \theta_\alpha \\ \mathrm{pr}\, v_{J\Theta}(j_{a\beta})\theta_\beta \wedge \eta_a &+ \mathrm{pr}\, v_{J\Theta}(j_{ab})\eta_b \wedge \eta_a]\, \mathrm{d}x = 0. \end{aligned} \tag{19}$$

On the other hand, in general,

$$\mathrm{pr}\, v_{J\Theta}\left(J^{(2)}_{AB}\right) = \sum_{E,F,k} \partial_x^k\left(J^{(2)}_{EF}\Theta_F\right)\frac{\partial}{\partial\left(\partial_x^k\phi_E\right)}\left(J^{(2)}_{AB}\right). \tag{20}$$

Here $k = 0, 1$. Furthermore,

$$\begin{aligned} \mathrm{pr}\, v_{J\Theta}\left(J^{(2)}_{AB}\right) = \sum_k \Bigg\{ &\partial_x^k(j_{\lambda\rho}\theta_\rho)\frac{\partial}{\partial\left(\partial_x^k u_\lambda\right)}\left(J^{(2)}_{AB}\right) + \partial_x^k(j_{\lambda e}\eta_e)\frac{\partial}{\partial\left(\partial_x^k u_\lambda\right)}\left(J^{(2)}_{AB}\right) \\ &+ \partial_x^k(j_{d\rho}\theta_\rho)\frac{\partial}{\partial\left(\partial_x^k \xi_d\right)}\left(J^{(2)}_{AB}\right) + \partial_x^k(j_{de}\eta_e)\frac{\partial}{\partial\left(\partial_x^k \xi_d\right)}\left(J^{(2)}_{AB}\right)\Bigg\}. \end{aligned} \tag{21}$$

Introducing

$$\begin{aligned} C_{\alpha\beta\lambda} &= C_{\beta\alpha\lambda} \\ \Omega_{ab\lambda} &= \Omega_{ba\lambda} \\ \Omega_{ab\lambda}K_{\lambda cd} &= \Omega_{ac\lambda}K_{\lambda bd} \end{aligned} \tag{22}$$

and using equation (21) in (18), we finally obtain

$$\begin{aligned}
\mathrm{pr}\, v_{J\Theta}(I) = \frac{1}{2}\int \{ & C_{\alpha\beta\lambda} b_{\lambda\rho}(\theta_{\rho,xxx}\wedge\theta_\beta\wedge\theta_\alpha - 2\theta_{\rho,xx}\wedge\theta_{\beta,xx}\wedge\theta_\alpha - 2\theta_{\rho,xx}\wedge\theta_{\beta,x}\wedge\theta_{\alpha,x}) \\
& + C_{\alpha\beta\lambda} C_{\lambda\rho\gamma}(u_\gamma\theta_{\rho,x}\wedge\theta_\beta\wedge\theta_\alpha + u_{\gamma,x}\theta_{\rho,}\wedge\theta_\beta\wedge\theta_\alpha \\
& + 4u_\gamma\theta_{\rho,x}\wedge\theta_{\beta,x}\wedge\theta_\alpha + 2u_{\gamma,x}\theta_\rho\wedge\theta_{\beta,x}\wedge\theta_\alpha) \\
& + M_{c\beta b}(L_{\alpha ac} + M_{a\alpha c} - N_{a\alpha c})\xi_b\eta_a\wedge\theta_{\beta,x}\wedge\theta_{\alpha,x} \\
& - N_{c\beta b}(K_{\alpha ac} - L_{\alpha ac} + N_{a\alpha c})\xi_{b,x}\eta_{a,x}\wedge\theta_\beta\wedge\theta_\alpha \\
& + [2C_{\alpha\beta\gamma}K_{\gamma ba} - M_{c\beta a}(K_{\alpha bc} - L_{\alpha bc} + N_{b\alpha c})]\xi_a\eta_{b,x}\wedge\theta_{\beta,x}\wedge\theta_\alpha \\
& + [2C_{\alpha\beta\gamma}L_{\gamma ba} - N_{c\alpha a}(M_{b\beta c} - N_{b\beta c} + L_{\beta bc})]\xi_{a,x}\eta_b\wedge\theta_{\beta,x}\wedge\theta_\alpha \\
& + [\Lambda_{ca}(K_{\alpha bc} + M_{b\alpha c}) - 6\Omega_{ba\lambda}b_{\lambda\alpha}]\eta_{a,xx}\wedge\eta_{b,x}\wedge\theta_\alpha \\
& + [\Lambda_{ca}(M_{b\alpha c} - N_{b\alpha c} + L_{\alpha bc}) - 2\Omega_{ba\lambda}b_{\lambda\alpha}]\eta_{a,xxx}\wedge\eta_b\wedge\theta_\alpha \\
& + \left[2\Omega_{ab\lambda}C_{\lambda\alpha\beta} + \tfrac{1}{2}\Omega_{ca\beta}(N_{b\alpha c} - 4L_{\alpha bc}) - \Omega_{cb\beta}M_{a\alpha c}\right]u_\beta\eta_a\wedge\eta_b\wedge\theta_{\alpha,x} \\
& + \left[\Omega_{ab\lambda}C_{\lambda\alpha\beta} - \tfrac{1}{2}\Omega_{ca\beta}(N_{b\alpha c} - 2L_{\alpha bc})\right]u_{\beta,x}\eta_a\wedge\eta_b\wedge\theta_\alpha \\
& + \tfrac{1}{3}\Omega_{cb\lambda}[3L_{\lambda ad} - K_{\lambda ad}]\xi_{d,x}\eta_a\wedge\eta_b\wedge\eta_c\}\,\mathrm{d}x = 0.
\end{aligned} \tag{23}$$

As can easily be seen, there is a trivial solution for equation (23) in which all constant coefficients vanish. There exists a non-trivial solution

$$\begin{aligned}
& b_{\alpha\lambda}C_{\lambda\beta\gamma} = b_{\beta\lambda}C_{\lambda\alpha\gamma} \qquad C_{\alpha\beta\lambda}C_{\lambda\gamma\rho} = C_{\alpha\gamma\lambda}C_{\lambda\beta\rho} \qquad K_{\lambda ab} = M_{a\lambda b} \\
& K_{\lambda ab} = 3L_{\lambda ab} \qquad 2M_{a\lambda b} = 3N_{a\lambda b} \qquad \Lambda_{bc}M_{a\alpha c} = 3b_{\alpha\beta}\Omega_{ab\beta} \\
& M_{c\alpha b}K_{\beta ac} = M_{c\beta b}K_{\alpha ac} \qquad C_{\alpha\beta\gamma}K_{\gamma ab} = K_{\alpha ac}K_{\beta bc}.
\end{aligned} \tag{24}$$

Thus $J^{(2)}_{AB}$ becomes a Hamilton operator with the set of equations (24). It describes the second Poisson structure. For KdV equation one can easily show that the sum of two Hamilton operators of bi-Hamiltonian structure is also a Hamilton operator because one of the Hamilton operators ($J = \partial_x$) trivially satisfies Jacobi identity [16]. In our case $J^{(1)}_{AB} + J^{(2)}_{AB}$ satisfies the graded Jacobi identity with the condition

$$\Omega_{ab\beta} - M_{a\beta b} - \frac{1}{2}(K_{\beta ab} + L_{\beta ab} - N_{a\beta b}) = 0 \tag{25}$$

and using our solution (24), equation (25) becomes

$$\Omega_{ab\beta} + N_{a\beta b} = 2K_{\beta ab}. \tag{26}$$

Furthermore, we obtain

$$\Omega_{ab\beta} = 4L_{\beta ab}. \tag{27}$$

Thus the Hamilton operators $J^{(1)}_{AB}$ and $J^{(2)}_{AB}$ constitute a super Hamiltonian pair. We can now rewrite the second operator in terms of $L_{\lambda ab}$ as

$$J^{(2)}_{AB} = \begin{pmatrix} b_{\alpha\beta}\partial_x^3 + C_{\alpha\beta\gamma}(u_\gamma\partial_x + \partial_x u_\gamma) & L_{\alpha bc}(2\xi_c\partial_x + \partial_x\xi_c) \\ L_{a\beta c}(\xi_c\partial_x + 2\partial_x\xi_c) & \Lambda_{ab}\partial_x^2 + 4L_{\lambda ab}u_\lambda \end{pmatrix}. \tag{28}$$

However, equations (24) and (25) provide information about algebra related to the evolution equations. In section 3 we shall derive the corresponding evolution equations that are coupled to the multi-component super KdV equations.

## 3. The multi-component super KdV systems

Bi-Hamiltonian formalism suggests the existence of infinitely many conserved quantities $\{H_k\}$ satisfying the recursion relation

$$\sum_B J^{(2)}_{AB} E_B(H_{k-1}) = \sum_B J^{(1)}_{AB} E_B(H_k) \tag{29}$$

where $k = 1, 2, 3, \ldots$. These infinitely many conserved quantities provide an extension of super KdV hierarchy to multi-component super KdV hierarchy. We now introduce the first two conserved quantities to obtain the first member of evolution equations as

$$H_0 = \frac{1}{2}\int [-\delta_{\alpha\beta}u_\alpha u_\beta + \delta_{ab}\xi_a\xi_{b,x}]\,\mathrm{d}x \tag{30}$$

$$H_1 = \frac{1}{2}\int [-b_{\alpha\beta}u_{\alpha,x}u_{\beta,x} + C_{\alpha\beta\gamma}u_\alpha u_\beta u_\gamma - \Lambda_{ab}\xi_{a,x}\xi_{b,xx} + 2K_{\alpha ab}u_\alpha\xi_a\xi_{b,x}]\,\mathrm{d}x. \tag{31}$$

Then one can easily derive integrable super coupled integrable evolution equations, which admit infinitely many conserved quantities due to the recursion relations (29), by using

$$\partial_t\phi_A = \sum_B J^{(1)}_{AB} E_B(H_1) = \sum_B J^{(2)}_{AB} E_B(H_0). \tag{32}$$

In this way we get the new class of integrable multi-component super KdV equations by using equations (24) and (26) as follows:

$$u_{\alpha,t} = b_{\alpha\beta}u_{\beta,xxx} + 3C_{\alpha\beta\gamma}u_{\beta,x}u_\gamma + K_{\alpha ab}\xi_a\xi_{b,xx} \tag{33}$$

$$\xi_{a,t} = \Lambda_{ab}\xi_a + K_{\lambda ab}(\xi_b u_{\lambda,x} + 2u_\lambda\xi_{b,x}). \tag{34}$$

In the bosonic limit when the fermionic variables vanish, the system reduces to multi-component KdV systems, known as degenerate Svinolupov system, in which $b_{\alpha\beta}$ is non-diagonalizable [4,5]. In this case, one-component limit is the KdV equation. Furthermore, if we choose the coefficients $b_{11} = -1$, $\Lambda_{11} = -4$, $C_{111} = 2$ and $K_{111} = 3$ satisfying the constraint equations (24) and variables $u_1 = u$ and $\xi_1 = \xi$, equations (33) and (34) become

$$u_t = -u_{xxx} + 6uu_x + 3\xi\xi_{xx} \tag{35}$$

$$\xi_t = -4\xi_{xxx} + 6u\xi_x + 3u_x\xi. \tag{36}$$

These are super KdV equations given in references [10, 11]. In other words, our equations reduce to one of the known one-component super KdV equations which consists of one bosonic and one fermionic variables.

## 4. The multi-component super mKdV systems

In this section we first introduce a super extension of Miura transformation using the notation of previous sections. The multi-component super Miura transformation is

$$u_\alpha = v_{\alpha,x} + \frac{1}{2}C_{\alpha\beta\gamma}v_\beta v_\gamma + K_{\alpha bc}\varepsilon_b\varepsilon_c \tag{37}$$

$$\xi_a = \varepsilon_{a,x} + \frac{1}{3}M_{a\lambda b}v_\lambda\varepsilon_b \tag{38}$$

where $v(x,t)$ and $\varepsilon(x,t)$ are new bosonic and fermionic variables, respectively. The multi-component super mKdV equations can be obtained from the multi-component super KdV equations by the multi-component super Miura transformations. This implies that any solution of the multi-component super mKdV equations gives a solution of the multi-component super KdV equations through the multi-component super Miura transformations.

When we substitute the transformation (37) into equation (33), we get the multi-component super mKdV equations

$$v_{\alpha,t} = -v_{\alpha,xxx} + \frac{3}{2}C_{\alpha\beta\gamma}C_{\beta\lambda\rho}v_\lambda v_\rho v_{\gamma,x} + \frac{1}{8}K_{\beta mn}C_{\alpha\beta\gamma}(2v_{\gamma,x}\varepsilon_m\varepsilon_{n,x} + v_\gamma\varepsilon_m\varepsilon_{n,xx}) + \frac{1}{4}K_{\alpha mn}\varepsilon_{n,x}\varepsilon_{m,xx} \quad (39)$$

$$\varepsilon_{a,t} = -4\varepsilon_{a,xxx} - K_{\beta ab}(v_{\beta,xx}\varepsilon_b + 2v_{\beta,x}\varepsilon_{b,x}) + K_{\beta ab}C_{\beta\lambda\rho}(v_\lambda v_\rho\varepsilon_{b,x} + v_{\lambda,x}v_\rho\varepsilon_b) \quad (40)$$

by employing the constraints (24) on the coefficients. As in the case of multi-component super KdV equations, equations (39) and (40) reduce to

$$v_t = \partial_x\left(2v^3 - v_{xx} + \frac{3}{4}\varepsilon\varepsilon_{xx} + \frac{3}{2}v\varepsilon\varepsilon_x\right) \quad (41)$$

$$\varepsilon_t = -4\varepsilon_{xxx} + (6vv_x - 3v_{xx})\varepsilon + 6\left(v^2 - v_x\right)\varepsilon_x \quad (42)$$

in the one-component limit by taking the coefficients as $C_{111} = 2$, $K_{111} = 3$ and the variables as $v_1 = v$ and $\varepsilon_1 = \varepsilon$. This is the super extension of the mKdV equation given by Kuperschmidt [10].

## 5. Conclusions

In this work we have found a new class of integrable multi-component super KdV equations. It is shown that they are bi-super Hamiltonian. The graded Jacobi identity associated with the Poisson structure defined by super Hamiltonian operators is satisfied by imposing constraints (24) on the coefficients introduced in the super Hamiltonian operators. These constraints could be important to describe the structure associated with our evolution equations. It is natural to expect that such relations would also imply the existence of generalized symmetries. Furthermore, by introducing a super Miura transformation a super extension of multi-component mKdV equations is obtained. This system also possesses the structure described by the constraints (24). We have shown that our equations are reduced to the well known one-component super equations and multi-component and one-component bosonic equations.

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